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On a Class of Multivariate Multisample Rank Order Tests II. Tests for Homogeneity of Dispersion Matrices

Madan Lal Puri and Pranab Kumar Sen

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ON A CLASS OF MULTIVARIATE MULTISAMPLE RANK ORDER TESTS II. TESTS FOR HOMOGENEITY OF DISPERSION MATRICES

Madan Lal Puri[1] and Pranab Kumar Sen[2]

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[1] This report represents results obtained in part while this author was at the University of North Carolina, Chapel Hill.

[2] Presently at the University of North Carolina, Chapel Hill. On leave of absence from the University of Calcutta.

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SUMMARY. In continuation of the authors' paper (1966), here a class of rank order tests for the homogeneity of several dispersion matrices is proposed and their properties are studied.

1. INTRODUCTION

The rank permutation technique in multivariate analysis is of quite recent origin, and is mostly confined to the problems of location and scale, and of association patterns (see Puri and Sen (1965) and Sen (1966) for references). But the authors are not aware of any nonparametric test for homogeneity of dispersion matrices. The object of the present investigation is to develop some nonparametric procedures for the problem of testing homogeneity of several dispersion matrices through the use of a class of rank order statistics which are extensions of Chernoff and Savage (1958) type of statistics.

In the parametric theory, the problem of testing homogeneity of dispersion matrices has nothing to do with the homogeneity of location parameters. However, in the nonparametric case the formulation of the hypothesis of invariance deems the identity of distributions, and this, in turn requires the homogeneity of location parameters. Thus we consider the following problems: (i) Tests for the homogeneity of dispersion matrices assuming the identity of location parameters. Some permutationally distribution free tests for this problem are considered here and their large sample properties are studied; (ii) Tests for the homogeneity of dispersion matrices without assuming the identity of location parameters. Here the basic permutation argument is not tenable, and only large sample properties are considered. Finally (iii) the problem of testing simultaneously the homogeneity of location parameters as well as the dispersion matrices is briefly discussed.

In the parametric case, tests for the homogeneity of dispersion matrices are based on the sample covariance matrices [see Anderson (1958, Chapter 10)]. However, in the nonparametric set up, the sample covariance matrix is not usually looked upon as a suitable statistic (matrix) because of its sensitiveness to outlying observations, of the necessity of the existence of the moments of the parent distribution, and of the fact that it may not retain its sufficiency or optimality for non-normal distributions. Due to these reasons, we shall formulate a class of regular functionals which seems desirable for nonparametric procedures. The corresponding rank order estimates are then used to formulate the proposed tests.

2. PRELIMINARY NOTIONS

Let $\underset{\sim}{X}_{\alpha}^{(k)} = (X_{1\alpha}^{(k)}, \ldots, X_{p\alpha}^{(k)})$, $\alpha = 1, \ldots, n_k$ be n_k independent and identically distributed (vector valued) random variables (i·i·d·r·v·) having a p-variate continuous cumulative distribution function (cdf) $F^{(k)}(\underset{\sim}{x})$, for $k = 1, \ldots, c (\geqq 2)$; all these c samples being assumed to be mutually independent. Denote $N = \sum_{k=1}^{c} n_k$ and assume that for all N, the inequalities

$$0 < \lambda_o \leqq \lambda_N^{(k)} = n_k/N \leqq 1 - \lambda_o < 1 \ , \quad \text{for } k = 1, \ldots, c \qquad (2.1)$$

hold for some fixed $\lambda_o \leqq 1/c$.

Let $F_{(i)}^{(k)}(x)$ and $F_{(i,j)}^{(k)}(x,y)$ be the marginal cdfs of $X_{i\alpha}^{(k)}$ and $(X_{i\alpha}^{(k)}, X_{j\alpha}^{(k)})$ respectively for $i < j = 1,\ldots,p$, and $k = 1,\ldots,c$. Let $\mathcal{F}_p$ be the class of all continuous p-variate cdfs for $p \geqq 1$. Then, we assume that

$$F^{(k)}(\underset{\sim}{x}) \in \mathcal{F}_p \Rightarrow F_{(i)}^{(k)}(x) \in \mathcal{F}_1 \text{ and } F_{(i,j)}^{(k)}(x,y) \in \mathcal{F}_2 \ . \quad (2.2)$$

Denote

$$H_{(i)}(x) = \sum_{k=1}^{c} \lambda_N^{(k)} \ F_{(i)}^{(k)}(x) \ . \quad (2.3)$$

Thus $H_{(i)}(x)$ is the combined population cdf of the ith variate for $i = 1,\ldots,p$. Let $J_{(i)}(u)$ for $i = 1,\ldots,p$ be some absolutely continuous function of u, defined in the open interval (0,1) and suppose that $J_{(i)}(u)$ is normalized in the following manner:

$$\int_0^1 J_{(i)}(u)\, du = 0 \ , \quad \int_0^1 J_{(i)}^2(u)\, du = 1; \ i = 1,\ldots,p \quad (2.4)$$

Later on we shall impose certain regularity conditions on $J_{(i)}(u)$. Let us define

$$Y_{i\alpha}^{(k)} = J_{(i)}\left[H_{(i)}(X_{i\alpha}^{(k)})\right] \ ; \quad i = 1,\ldots,p \ ; \ k = 1,\ldots,c. \quad (2.5)$$

and we shall call $Y_{i\alpha}^{(k)}$ as the generalized grade of $X_{i\alpha}^{(k)}$. We shall employ these generalized grades to formulate the desired class of tests.

Let us now define

$$\theta_{ij}(H) = \operatorname{cov}\left[(J_{(i)}[H_{(i)}(x)],\ J_{(j)}[H_{(j)}(y)]) \middle| H(\underset{\sim}{x})\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{(i)}\left[H_{(i)}(x)\right] J_{(j)} \left[H_{(j)}(y)\right] dH_{(i,j)}(x,y) \qquad (2.6)$$

where

$$H_{(i,j)}(x,y) = \sum_{k=1}^{c} \lambda_N^{(k)} F_{(i,j)}^{(k)}(x,y) \quad \text{for } i \neq j = 1,\dots,p \qquad (2.7)$$

and $H_{(i)}(x)$ is as defined in (2.3). Also denote

$$\theta_{ij}^{(k)} = \operatorname{cov}\left[(J_{(i)}[H_{(i)}(x)],\ J_{(j)}[H_{(j)}(y)]) \middle| F^{(k)}(\underset{\sim}{x})\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{(i)}\left[H_{(i)}(x)\right] J_{(j)} \left[H_{(j)}(y)\right] dF_{(i,j)}^{(k)}(x,y) - \mu_{N,i}^{(k)}\mu_{N,j}^{(k)} \qquad (2.8)$$

where

$$\mu_{N,i}^{(k)} = \int_{-\infty}^{\infty} J_{(i)}\left[H_{(i)}(x)\right] dF_{(i)}^{(k)}(x); \quad i = 1,\dots,p\ ;\ k=1,\dots,c \qquad (2.9)$$

Finally let

$$\underset{\sim}{\textcircled{H}}^{*} = ((\theta_{ij}(H)))_{i,j=1,\ldots,p} \quad \text{and} \quad \underset{\sim}{\textcircled{H}}^{(k)} = ((\theta_{ij}^{(k)}))_{i,j=1,\ldots,p}; \qquad k=1,\ldots,c. \quad (2.10)$$

Thus, $\underset{\sim}{\textcircled{H}}^{(k)}$ is the dispersion matrix of the generalized grades of the kth sample, $k = 1,\ldots,c$. The purpose of introducing the dispersion matrices in (2.10) is to work with a class of dispersion matrices which are invariant under certain transformations on the variables, are less sensitive to outlying observations, and at the same time, are reasonably informative measures. In particular, if $J_{(i)}(u)$ is monotone in u $(0 < u < 1)$ for all $i = 1,\ldots,p$, $\underset{\sim}{\textcircled{H}}^{(k)}$, $k = 1,\ldots,c$ defined in (2.10) will be invariant under monotone transformations of the co-ordinate variables.

Now since the cdfs $F^{(1)},\ldots,F^{(c)}$ are all unknown, so are also the $(c+1)$ matrices defined in (2.10). To estimate these we follow Von Mises (1947) approach, and define for any $q \leqq p$

$$F_{N(i_1,\ldots,i_q)}^{(k)}(x_1,\ldots,x_q) = \frac{1}{n_k}\,[\text{number of}\,(X_{i_1\alpha}^{(k)},\ldots,X_{i_q\alpha}^{(k)}) \leqq (x_1,\ldots,x_q)\ ,\ \alpha = 1,\ldots,n_k] \quad (2.11)$$

$$H_{N(i_1,\ldots,i_q)}(x_1,\ldots,x_q) = \sum_{k=1}^{c} \lambda_N^{(k)}\ F_{N(i_1,\ldots,i_q)}^{(k)}(x_1,\ldots,x_q) \quad (2.12)$$

for all $i_1 \neq \ldots \neq i_q = 1,\ldots,p$; $q = 1,\ldots,p$. Again, let

$$F^{(k)}_{(i_1,\ldots,i_q)}(x_1,\ldots,x_q) = P\{X^{(k)}_{i_j\alpha} \leqq x_j \ ; \ j = 1,\ldots,q\} \quad (2.13)$$

$$H_{(i_1,\ldots,i_q)}(x_1,\ldots,x_q) = \sum_{k=1}^{c} \lambda_N^{(k)} \ F^{(k)}_{(i_1,\ldots,i_q)}(x_1,\ldots,x_q) \quad (2.14)$$

for all $i_1 \neq \ldots \neq i_q = 1,\ldots,p$; $q = 1,\ldots,p$.

Now in actual practice, we have a sequence of functions $J_{N(i)}(u)$ which converges to $J_{(i)}(u)$ as $N \longrightarrow \infty$ for all $0 < u < 1$, and $i = 1,\ldots,p$. Thus we may estimate $\theta_{ij}(H)$ defined in (2.6) by

$$T^*_{N,ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{N(i)}\left[H_{N(i)}(x)\right] \ J_{N(j)} \ \left[H_{N(j)}(y)\right] \ dH_{N(i,j)}(x,y)$$

$$= \frac{1}{N} \ \sum_{k=1}^{c}\sum_{\alpha=1}^{n_k} J_{N(i)}\left[H_{N(i)}(X^{(k)}_{i\alpha})\right] \ J_{N(j)} \ \left[H_{N(j)}(X^{(k)}_{j\alpha})\right] \quad (2.15)$$

for all $i,j = 1,\ldots,p$. Since $T^*_{N,ij} = T^*_{N,ji}$, the number of statistics $T^*_{N,ij}$ is actually $p(p+1)/2$. Now let us denote

$$\underset{\sim}{T}^*_N = \{T^*_{N,ij} \ , \ i \leqq j = 1,\ldots,p\} \quad (2.16)$$

$$T^{(k)}_{N,i} = \int_{-\infty}^{\infty} J_{N(i)}\left[H_{N(i)}(x)\right] \ dF^{(k)}_{N(i)}(x) \ ; \quad i = 1,\ldots,p; \quad (2.17)$$

$$k = 1,\ldots,c,$$

$$T_{N,ij}^{(k)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{N(i)}\left[H_{N(i)}(x)\right] J_{N(j)} \left[H_{N(j)}(y)\right] dF_{N(i,j)}^{(k)}(x,y)$$

$$- T_{N,i}^{(k)} T_{N,j}^{(k)}$$

$$= \frac{1}{n_k} \sum_{\alpha=1}^{n_k} J_{N(i)}\left[H_{N(i)}(X_{i\alpha}^{(k)})\right] J_{N(j)} \left[H_{N(j)}(X_{j\alpha}^{(k)})\right]$$

$$- T_{N,i}^{(k)} T_{N,j}^{(k)} \qquad (2.14)$$

and, finally

$$\underset{\sim}{T}_N^{(k)} = \left\{T_{N,ij}^{(k)}, \quad i \leqq j = 1,\ldots,p\right\}; \quad k = 1,\ldots,c \,. \qquad (2.19)$$

Then, our proposed tests are based on some functions (see (3.24)) of $\underset{\sim}{T}_N^{(k)}$, $k = 1,\ldots,c$ and $\underset{\sim}{T}_N^{*}$.

It may be noted that the class of dispersion matrices considered above contains many well known nonparametric measures as special cases. For example, if we let

$$J_{N(i)}\ (\alpha/(N+1)) = \left[12/(N^2-1)^{1/2}\right](\alpha - \frac{N+1}{2}) \ , \ \alpha = 1,\ldots,N;$$

$i = 1,\ldots,p$; then $\underset{\sim}{T}_N^{*}$ reduces to the Spearman's rank correlations [cf. Kendall (1955)], while for $J_{(i)}(u) = \sqrt{12}\ (u - 1/2)$, $\underset{\sim}{\textcircled{H}}^{*}$ reduces to the grade correlations matrix. The nonparametric measures by Blomquist (1950) and Bhuchongkul (1964) among some others belong to this class.

We now consider the statistical formulations of the problem of testing the homogeneity of dispersion matrices.

First we consider equality of covariance matrices, assuming the identity of locations. In such a case our null hypothesis is

$$H_o^{(1)}: \ F^{(1)}(\underset{\sim}{x}) = \ldots = F^{(c)}(\underset{\sim}{x}) \qquad (2.20)$$

(which implies that $\underset{\sim}{\textcircled{H}}^{(1)} = \ldots = \underset{\sim}{\textcircled{H}}^{(c)}$), against the set of alternatives that $\underset{\sim}{\textcircled{H}}^{(1)},\ldots,\underset{\sim}{\textcircled{H}}^{(c)}$ are not all identical. We may note that $H_o^{(1)}$ will include test for identity of scales and of association patterns of the c cdfs. Next we consider the equality of dispersion matrices without assuming the identity of locations. In such a case, denoting

$$F^{(k)}(\underset{\sim}{x}) = F_o^{(k)}(\underset{\sim}{x} - \underset{\sim}{\delta}^{(k)}) \ , \ k = 1,\ldots,c \qquad (2.21)$$

where $\underset{\sim}{\delta}^{(k)} = (\delta_1^{(k)},\ldots,\delta_p^{(k)})$; $k = 1,\ldots,c$ are c real p-vectors, we test

$$H_o^{(2)} = F_o^{(1)}(\underset{\sim}{x}) = \ldots = F_o^{(c)}(\underset{\sim}{x}) \qquad (2.22)$$

against similar type of alternatives involving $\underset{\sim}{\textcircled{H}}^{(1)},\ldots,\underset{\sim}{\textcircled{H}}^{(c)}$, where of course, in the definition of $\underset{\sim}{\textcircled{H}}^{(k)}$'s ; $F^{(1)},\ldots,F^{(c)}$ are replaced by $F_o^{(1)},\ldots,F_o^{(c)}$ respectively. Finally, we consider the hypothesis that both locations and dispersion matrices are the same; that is, we test

$$H_o^{(3)}: \quad F^{(1)}(\underset{\sim}{x}) = \ldots = F^{(c)}(\underset{\sim}{x}) \tag{2.23}$$

where $F^{(k)}(\underset{\sim}{x})$, $k = 1,\ldots,c$ is given by (2.21), against the alternatives that at least one of the following two inequalities is not true.

$$\underset{\sim}{\delta}^{(1)} = \ldots = \underset{\sim}{\delta}^{(c)} \; ; \quad \underset{\sim}{\textcircled{H}}^{(1)} = \ldots = \underset{\sim}{\textcircled{H}}^{(c)} \; . \tag{2.24}$$

3. PERMUTATION TESTS FOR $H_o^{(1)}$

Let us pool the $N = \sum_{k=1}^{c} n_k$ observations $\underset{\sim\alpha}{X}^{(k)}$, $\alpha = 1,\ldots,n_k$; $k = 1,\ldots,c$ into a combined sample, and denote the sample point by $p \times N$ matrix

$$Z_N' = (\underset{\sim 1}{X}^{(1)'},\ldots,\underset{\sim n_c}{X}^{(c)'}) \tag{3.1}$$

and the sample space by $\mathcal{Z}_N$. Under $H_o^{(1)}$, the joint distribution of Z_N remains invariant under the finite group of $N!$ permutations of the N columns of Z_N'. Hence, conditionally given Z_N, all the $N!$ sample points generated by these $N!$ permutations of the columns of $\underset{\sim N}{Z}'$ are equiprobable, each having the conditional probability $1/N!$. Let us denote this conditional probability measure (defined over the $N!$ equiprobable realizations) by $\mathcal{P}_N$. Now, ranking the N elements in each row of $\underset{\sim N}{Z}'$ in increasing order of magnitude, we obtain a p x N matrix

$$R_N = \begin{pmatrix} R^{(1)}_{11} & \cdots & R^{(1)}_{1n_1} & \cdots & R^{(c)}_{1n_c} \\ \vdots & & \vdots & & \vdots \\ R^{(1)}_{p1} & \cdots & R^{(1)}_{pn_1} & \cdots & R^{(c)}_{pn_c} \end{pmatrix} \tag{3.2}$$

where by virtue of continuity of the cdfs, the possibility of ties may be ignored in probability. For every $i(= 1,\ldots,p)$, replacing the ranks $\alpha(= 1,\ldots,N)$ in the ith line of R_N by a set of general scores $\{E^{(i)}_{N,\alpha} = J_{N(i)}(\alpha/N),\ \alpha = 1,\ldots,N\}$, we get the corresponding score matrix

$$E_N = \begin{pmatrix} E^{(1)}_{N,R^{(1)}_{11}} & \cdots & E^{(1)}_{N,R^{(1)}_{1n_1}} & \cdots & E^{(c)}_{N,R^{(c)}_{1n_c}} \\ \vdots & & \vdots & & \vdots \\ E^{(p)}_{N,R^{(1)}_{p1}} & \cdots & E^{(p)}_{N,R^{(1)}_{pn_1}} & \cdots & E^{(p)}_{N,R^{(c)}_{pn_c}} \end{pmatrix} \tag{3.3}$$

With this terminology, we can write $T^{(k)}_{N,ij}$ and $T^{*}_{N,ij}$ defined by (2.18) and (2.15) as

$$T^{(k)}_{N,ij} = \frac{1}{n_k}\sum_{\alpha=1}^{n_k} E^{(i)}_{N,R^{(k)}_{i\alpha}}\, E^{(j)}_{N,R^{(k)}_{j\alpha}} - \left[\frac{1}{n_k}\sum_{\alpha=1}^{n_k} E^{(i)}_{N,R^{(k)}_{i\alpha}}\right]\left[\frac{1}{n_k}\sum_{\alpha=1}^{n_k} E^{(j)}_{N,R^{(k)}_{j\alpha}}\right] \tag{3.4}$$

$i \leqq j = 1,\ldots,p;\ k = 1,\ldots,c$

$$T^{*}_{N,ij} = \frac{1}{N}\sum_{k=1}^{c}\sum_{\alpha=1}^{n_k} E^{(i)}_{N,R^{(k)}_{i\alpha}}\; E^{(j)}_{N,R^{(k)}_{j\alpha}} \;;\; i \leqq j = 1,\ldots,p \;. \qquad (3.5)$$

Operationally, it will be much simpler to work with the slightly adjusted statistics:

$$S^{(k)}_{N,ij} = \frac{1}{n_k-1}\left\{\sum_{\alpha=1}^{n_k} E^{(i)}_{N,R^{(k)}_{i\alpha}}\; E^{(j)}_{N,R^{(k)}_{j\alpha}} - \frac{1}{n_k}\left[\sum_{\alpha=1}^{n_k} E^{(i)}_{N,R^{(k)}_{i\alpha}}\right]\left[\sum_{\alpha=1}^{n_k} E^{(j)}_{N,R^{(k)}_{j\alpha}}\right]\right\} \qquad (3.6)$$

$$i \leqq j = 1,\ldots,p;\; k = 1,\ldots,c \;,$$

$$S^{*}_{N,ij} = \frac{1}{N-1}\left\{\sum_{k=1}^{c}\sum_{\alpha=1}^{n_k} E^{(i)}_{N,R^{(k)}_{i\alpha}}\; E^{(j)}_{N,R^{(k)}_{j\alpha}} - N\,\bar{E}^{(i)}_{N}\,\bar{E}^{(j)}_{N}\right\} \qquad (3.7)$$

where

$$\bar{E}^{(i)}_{N} = \frac{1}{N}\sum_{k=1}^{c}\sum_{\alpha=1}^{n_k} E^{(i)}_{N,R^{(k)}_{i\alpha}} = \int_{0}^{1} J_{N(i)}\left[H_{N(i)}(x)\right]dH_{N(i)}(x), i=1,\ldots,p \qquad (3.8)$$

Note that

$$S^{(k)}_{N,ij} = \left[n_k/(n_k-1)\right]T^{(k)}_{N,ij}$$

and

$$S^{*}_{N,ij} = \left|N/N-1)\right|\left|T^{*}_{N,ij} - N\bar{E}^{(i)}_{N}\,\bar{E}^{(j)}_{N}\right| \;. \qquad (3.9)$$

Now under the permutational probability measure $\mathcal{P}_N$, there are N! equally likely (conditionally) permutations of the columns of $\underset{\sim}{R}_N$ (and hence of $\underset{\sim}{E}_N$). Hence, by an adaptation of the same argument as in Puri and Sen (1966), we have

$$E(S^{(k)}_{N,ij} \,|\, \mathcal{P}_N) = S^{*}_{N,ij} \quad \text{for all } i \leqq j = 1,\ldots,p \ .$$

Furthermore, $S^{(k)}_{N,ij}$ in (3.6) can also be expressed as a U-statistics in the scores $E^{(i)}_{N,R^{(k)}_{i\alpha}}$; $i = 1,\ldots,p$; $\alpha = 1,\ldots,n_k$ as follows:

$$S^{(k)}_{N,ij} = \frac{1}{2}\binom{n_k}{2}^{-1} \sum_{\alpha<\beta=1}^{n_k} (E^{(i)}_{N,R^{(k)}_{i\alpha}} - E^{(i)}_{N,R^{(k)}_{i\beta}})(E^{(j)}_{N,R^{(k)}_{j\alpha}} - E^{(j)}_{N,R^{(k)}_{j\beta}}) \tag{3.10}$$

$$\text{for } i,\ j = 1,\ldots,p;\ k = 1,\ldots,c \ .$$

Let us now define

$$4\zeta^{(1)}_{ij,i'j'}(\underset{\sim}{R}_N) =$$

$$\frac{1}{N(N-1)(N-2)} \sum_{k=1}^{c}\sum_{q=1}^{c}\sum_{r=1}^{c}\sum_{\alpha=1}^{n_k}\sum_{\beta=1}^{n_q}\sum_{\gamma=1}^{n_r} (E^{(i)}_{N,R^{(k)}_{i\alpha}} - E^{(i)}_{N,R^{(q)}_{i\beta}}) \cdot$$

$$(E^{(j)}_{N,R^{(k)}_{j\alpha}} - E^{(j)}_{N,R^{(q)}_{j\beta}}) \cdot (E^{(i')}_{N,R^{(k)}_{i'\alpha}} - E^{(i')}_{N,R^{(r)}_{i'\gamma}})(E^{(j')}_{N,R^{(k)}_{j'\alpha}} - E^{(j')}_{N,R^{(r)}_{j'\gamma}})$$

$$- S^{*}_{N,ij} \cdot S^{*}_{N,i'j'} \quad \text{for } i,\ i',\ j,\ j' = 1,\ldots,p; \tag{3.11}$$

$$4\zeta^{(2)}_{ij,i'j'}(\underset{\sim}{R}_N) =$$

$$\frac{1}{N(N-1)} \sum_{k=1}^{c} \sum_{q=1}^{c} \sum_{\alpha=1}^{n_k} \sum_{\beta=1}^{n_k} \left(E^{(i)}_{N,R^{(k)}_{i\alpha}} - E^{(i)}_{N,R^{(q)}_{i\beta}}\right)\left(E^{(j)}_{N,R^{(k)}_{j\alpha}} - E^{(j)}_{N,R^{(q)}_{j\beta}}\right) \cdot$$

$$\left(E^{(i')}_{N,R^{(k)}_{i'\alpha}} - E^{(i')}_{N,R^{(q)}_{i'\beta}}\right)\left(E^{(j')}_{N,R^{(k)}_{j'\alpha}} - E^{(j')}_{N,R^{(q)}_{j'\beta}}\right) - S^{*}_{N,ij}\, S^{*}_{N,i'j'} \qquad (3.12)$$

for i, j, i', j' = 1,...,p .

Then, by an adaptation of the results of Nandi and Sen (1963) and Sen (1966), it can be shown after some algebraic manipulations that

$$\operatorname{cov}\left\{S^{(k)}_{N,ij},\ S^{(q)}_{N,i'j'} \,\middle|\, \mathcal{P}_N\right\}$$

$$= \frac{\binom{n_k-2}{2}}{\binom{n_k}{2}} \sum_{d=1}^{2} \binom{2}{d} \left\{ \frac{\binom{n_k-2}{2-d}}{\binom{n_k-2}{2}} - \frac{\binom{N-2}{2-d}}{\binom{N-2}{2}} \right\} \zeta^{(d)}_{ij,i'j'}(\underset{\sim}{R}_N) \quad \text{if } k = q \ , \qquad (3.13)$$

$$= - \binom{N-2}{2}^{-1} \sum_{d=1}^{2} \binom{2}{d}\binom{N-2}{2-d}\, \zeta^{(d)}_{ij,i'j'}(\underset{\sim}{R}_N), \quad \text{if } k \neq q, \qquad (3.14)$$

for i, i', j, j' = 1,...,p; k,q = 1,...,c .

Thus, for small samples one may work with the random variables

$$\left\{S^{(k)}_{N,ij} - S^{*}_{N,ij}\ ;\ \ i \leqq j = 1,\dots,p,\ k = 1,\dots,c\right\}, \qquad (3.15)$$

and considering the generalized inverse of their (permutation) covariance matrix, arrive at the desired quadratic form as suitable test statistics. Under $\mathcal{P}_N$, this statistic will have only $N!/\prod_1^c n_k!$ equally likely (not necessarily all distinct) values, and hence, an exact size $\varepsilon (0 < \varepsilon < 1)$ test for $H_0^{(1)}$ can be constructed. However, the expressions in (3.11) through (3.14) are quite complicated. If N is not too small, it will be convenient to consider the following permutation test.

By virtue of (2.4), $\bar{E}_N^{(i)}$ defined in (3.8) converges to zero as $N \longrightarrow \infty$. In fact, (we shall see later on) $\bar{E}_N^{(i)} = o(N^{-1/2})$ for all $i = 1,\ldots,p$. Hence from (3.5) and (3.7)

$$\left| T^*_{N,ij} - \frac{N-1}{N} S^*_{N,ij} \right| = o(N^{-1/2}) . \qquad (3.16)$$

Further, from the results of Puri and Sen (1966), under $\mathcal{P}_N'$

$$\left| \frac{1}{n_k} \sum_{\alpha=1}^{n_k} E^{(i)}_{N,R^{(k)}_{i\alpha}} - \bar{E}^{(i)}_N \right| = O_p(N^{-1/2}) \qquad (3.17)$$

for all $i = 1,\ldots,p$; $k = 1,\ldots,c$. Consequently under $\mathcal{P}_N'$

$$\left| S^{(k)}_{N,ij} - \frac{1}{n_k} \sum_{\alpha=1}^{n_k} E^{(i)}_{N,R^{(k)}_{i\alpha}} \, E^{(j)}_{N,R^{(k)}_{j\alpha}} \right| = O_p(N^{-1}) \qquad (3.18)$$

for all $i, j = 1,\ldots,p$; $k = 1,\ldots,c$.

Now let us define

$$v_{ij,i'j'}(\underset{\sim}{R}_N) = \frac{1}{N} \sum_{k=1}^{c} \sum_{\alpha=1}^{n_k} E^{(i)}_{N,R^{(k)}_{i\alpha}} \; E^{(j)}_{N,R^{(k)}_{j\alpha}} \; E^{(i')}_{N,R^{(k)}_{i'\alpha}} \; E^{(j')}_{N,R^{(k)}_{j'\alpha}}$$

$$- S^{*}_{N,ij} \, S^{*}_{N,i'j'} \qquad (3.19)$$

for all i, i', j, j' = 1,...,p .

Setting

$$r = (i - 1)(2p - i)/2 + j \quad \text{for } i \leqq j = 1,\ldots,p \quad (3.20)$$

we rewrite

$$\left\{S^{(k)}_{N,ij} \, , \; i \leqq j = 1,\ldots,p\right\} \text{ as } \left\{S^{(k)}_{N,r} \, , \; r = 1,\ldots,p(p+1)/2\right\} = \underset{\sim}{S}^{(k)}_N \quad (3.21)$$

for k = 1,...,c ,

$$\left\{S^{*}_{N,ij} \, , \; i \leqq j = 1,\ldots,p\right\} \text{ as } \left\{S^{*}_{N,r} \, , \; r = 1,\ldots,p(p+1)/2\right\} = \underset{\sim}{S}^{*}_N \quad (3.22)$$

and

$$\underset{\sim}{V}_N(\underset{\sim}{R}_N) = ((v_{rs}(R_N)))_{r,s} = 1,\ldots,p(p+1)/2 \qquad (3.23)$$

where r is defined as in (3.20) with (i,j) replaced by (i',j').

Then, proceeding as in Puri and Sen (1966), we consider the statistic

$$\mathcal{L}_N = \sum_{k=1}^{c} n_k \left[\underset{\sim}{S}^{(k)}_N - \underset{\sim}{S}^{*}_N\right] V_N^{-1}(\underset{\sim}{R}_N) \left[S_N^{(k)} - \underset{\sim}{S}^{*}_N\right]' \qquad (3.24)$$

for the problem of testing $H_0^{(1)}$. (Here $V_N^{-1}(\underset{\sim}{R_N})$ is in the inverse of $V_N(\underset{\sim}{R_N})$, and is assumed to be positive definite). Thus, we have the following rule:

$$\begin{aligned} \text{if } \mathcal{L}_N &\geqq \mathcal{L}_{N,\varepsilon}\ , \text{ reject } H_0^{(1)} \\ &< \mathcal{L}_{N,\varepsilon}\ , \text{ accept } H_0^{(1)} \end{aligned} \tag{3.25}$$

where $\mathcal{L}_{N,\varepsilon}$ is so chosen that $P\{\mathcal{L}_N \geqq \mathcal{L}_{N,\varepsilon} \mid H_0^{(1)}\} = \varepsilon$,

where $0 < \varepsilon < 1$ is the preassigned level of significance of our test. In small smaples, we can evaluate $\mathcal{L}_{N,\varepsilon}$ by referring to the exact permutation distribution of $\mathcal{L}_N$ generated by the N! equally likely permutations of the columns of (3.3), while for large samples, we shall show in the next section that

$$\mathcal{L}_{N,\varepsilon} \xrightarrow{P} \chi^2_{\varepsilon,(c-1)p(p+1)/2} \tag{3.26}$$

where $\chi^2_{\varepsilon,r}$ is the $100(1-\varepsilon)$ %. point of a chidistribution with r degrees of freedom.

4. ASYMPTOTIC PERMUATION DISTRIBUTION OF $\mathcal{L}_N$

As in Section 3, we define

$$E_{N,\alpha}^{(i)} = J_{N(i)}(\alpha/N); \quad \alpha = 1,\dots,N; \quad i = 1,\dots.p\ . \tag{4.1}$$

and as in Chernoff Savage (1958), we extend the domain of $J_{N(i)}(u)$ to $(0,1)$ by letting $J_{N(i)}$ be constant on $[\alpha/N, (\alpha+1)/N)$. Furthermore, we shall make the following assumptions*:

Assumption I. $J_{(i)}(u) = \lim_{N\to\infty} J_{N(i)}(u)$ exists for $0 < u < 1$ and is not a constant (for $i = 1,\ldots,p$).

Assumption II.

$$\iint_{I_{N(i)}\cap I_{N(j)}} \Big\{J_{N(i)}\left[H_{N(i)}(x)\right] J_{N(j)} \left[H_{N(j)}(y)\right] - J_{(i)}\left[H_{N(i)}(x)\right] J_{(j)} \left[H_{N(j)}(y)\right]\Big\} \, dF^{(k)}_{(i,j)}(x,y) = o_p(N^{-1/2})$$

$$i,j = 1,\ldots,p;\ k = 1,\ldots,c \qquad (4.2)$$

where

$$I_{N(i)} = \left\{x:\ 0 < H_{N(i)}(x) < 1\right\};\ i = 1,\ldots,p\ .$$

Assumption III. $J_{N(i)} = o(N^{1/4})$. (4.3)

Assumption IV.

$$\left|J^{(r)}_{(i)}(u)\right| = \left|\frac{\partial^r}{\partial u^r} J_{(i)}(u)\right| \leqq K[u(1-u)]^{\alpha r} \qquad (4.4)$$

---

*The assumptions 1 to 4 are similar to those of Bhuchangkul (1964). These assumptions are somewhat more restrictive as compared to the ones in Puri and Sen (1966). However, this restriction may be ascribed to the fact that here we are dealing with rank order covariance statistics, while in the earlier paper, we dealt only with linear rank order statistics.

for $r = 0, 1, 2$, where $0 < \alpha_o < 1/8$; $\alpha_1 = -1$ and $\alpha_2 = -2$.

Now let us denote

$$\mu^{(k)}_{Nij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{(i)}\left[H_{(i)}(x)\right] J_{(j)}\left[H_{(j)}(y)\right] dF^{(k)}_{(i,j)}(x,y);\ k=1,\dots,c; \qquad (4.5)$$

$$\mu^{*}_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{(i)}\left[H_{(i)}(x)\right] J_{(j)}\left[H_{(j)}(y)\right] dH_{(i,j)}(x,y) \qquad (4.6)$$

$$\nu^{*}_{ij,i'j'} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{(i)}\left[H_{(i)}(x_i)\right] J_{(j)}\left[H_{(j)}(x_j)\right]$$

$$J_{(i')}\left[H_{(i')}(x_{i'})\right] J_{(j')}\left[H_{(j')}(x_{j'})\right]$$

$$dH_{(i,j,i',j')}(x_i,x_j,x_{i'},x_{j'})$$

$$-\ \mu^{*}_{ij}\,\mu^{*}_{i'j'} \quad \text{for } i,j,i',j'=1,\dots,p. \qquad (4.7)$$

Adopting the suffixing system of (3.20), we rewrite $\nu^{*}_{ij,i'j'}$ as ν^{*}_{rs} for $r,s = 1,\dots,p(p+1)/2$ and denote

$$\underset{\sim}{\nu}^{*} = ((\nu^{*}_{rs}))_{r,s\,=\,1,\dots,p(p+1)/2}\,. \qquad (4.8)$$

Assumption V. $\underset{\sim}{\nu}^{*}$ defined by (4.8) is positive definite.

Then, we have the following Theorems.

THEOREM 4.1. *Under assumptions* I *to* V, $\underset{\sim}{V}_N(\underset{\sim}{R}_N)$, *defined in* (3.23), *is equivalent in probability to* $\underset{\sim}{\nu}^*$, *defined in* (4.8), *as* $N \longrightarrow \infty$ *uniformly in* $\lambda_N^{(1)}, \ldots, \lambda_N^{(c)}$ *and all* $F^{(1)}, \ldots, F^{(c)} \in \mathcal{F}_p$.

The proof follows precisely along the same line as in Theorem 4.2 of Puri and Sen (1966) with straight forward extensions, and hence is omitted.

THEOREM 4.2. *Under assumptions* I to V, *the permutation distribution of* $\mathcal{L}_N$ *(defined by* (3.24)) *asymptotically reduces to a chisquare distribution with* (c-1)p(p+1)/2 *degrees of freedom*.

Proof. By virtue of (3.18), in $\mathcal{L}_N$ we can replace

$S_{N,ij}^{(k)}$ by $Z_{N,ij}^{(k)} = \frac{1}{n_k} \sum_{\alpha=1}^{n_k} E_{N,R_{i\alpha}}^{(i)} \; E_{N,R_{j\alpha}}^{(j)}$ for all $i \leqq j = 1, \ldots, p$;

$k = 1, \ldots, c$. We shall first prove that under $\mathcal{P}_N$,

$\{N^{1/2}(Z_{N,ij}^{(k)} - T_{N,ij}^*),\ i \leqq j = 1, \ldots, p; \quad k = 1, \ldots, c\}$ has asymptotically a (c-1)p(p+1)/2 variate normal distribution. For this, it suffices to show that for any non-null real $\underset{\sim}{\delta} = (\delta_1, \ldots, \delta_c)$, such that $\sum_{k=1}^{c} n_k \delta_k = 0$,

$\{N^{1/2} \sum_{k=1}^{c} \delta_k \, Z_{N,ij}^{(k)},\ i \leqq j = 1, \ldots, p\}$ has asymptotically (under $\mathcal{P}_N$) a multivariate normal distribution of rank $p(p+1)/2$. Now let us denote

$$\sum_{k=1}^{c} \delta_k \, Z^{(k)}_{N,ij} = \sum_{k=1}^{c} \delta_k \cdot \frac{1}{n_k} \sum_{\alpha=1}^{n_k} E^{(i)}_{N,R^{(k)}_{i\alpha}} \, E^{(j)}_{N,R^{(k)}_{j\alpha}}$$

$$= \sum_{k=1}^{c} \delta_k \cdot T^{(k)}_{N,r} = W_{N,r} \qquad (4.9)$$

where $r = \left[(i-1)(2p-i)/2\right] + j$ for $i \leqq j = 1,\ldots,p$ (cf. 3.20). and define

$$a^{(k)}_{N,r,\alpha} = E^{(i)}_{N,R^{(k)}_{i\alpha}} \, E^{(j)}_{N,R^{(k)}_{j\alpha}}, \text{ if } r = \left[(i-1)(2p-i)/2\right] + j \;, \qquad (4.10)$$

then, we have N vectors

$$\underset{\sim}{a}^{(k)\prime}_{N\alpha} = \left(a^{(k)}_{N,1,\alpha}, \ldots, a^{(k)}_{N,p(p+1)/2,\alpha}\right) \text{ for } \alpha = 1,\ldots,n_k; \qquad (4.11)$$
$$k = 1,\ldots,c,$$

and under $\mathcal{P}_N$, all possible N! permutations of these N vectors among themselves are (conditionally) equally likely, each permutation having common permutational probability 1/N! Thus $\underset{\sim}{W}_N = \left\{W_{N,r}: \; r = 1,\ldots,p(p+1)/2\right\}$ is a $p(p+1)/2$ vector where each element is a linear permutation statistic of the type considered by Wald-Wolfowitz-Noether-Hoeffding-Hájek among others. Thus by a direct application of

Theorem 7.1 of Hájek (1961), it follows that $\underset{\sim}{W}_N$ has asymptotically a p(p+1)/2 variate normal distribution. The rest of the proof follows by straight forward routine analysis. Hence the theorem.

By virtue of Theorem 4.2, $\mathcal{L}_{N,\varepsilon}$ defined in (3.25) reduces to $\chi^2_{\varepsilon,(c-1)p(p+1)/2}$ which asserts the truth of (3.26).

5. ASYMPTOTIC NORMALITY OF $\underset{\sim}{S}_N^{(k)}$, k = 1,...,c FOR ARBITRARY $F^{(1)},\ldots,F^{(c)}$

The main theorem of this section is the following.

Theorem 5.1. *Under the assumptions* I *to* V *of Section* 4, *the random variables* $N^{1/2}\left[U_{N,ij}^{(k)} - \mu_{N,ij}^{(k)} - \mu_{N,i}^{(k)}\mu_{N,j}^{(k)},\ T_{N,i}^{(k)} - \mu_{N,i}^{(k)};\ i \leq j = 1,\ldots,p\right]$ *where* $T_{N,i}^{(k)}$ *and* $\mu_{N,ij}^{(k)}$ *are given by* (2.16), (2.8) *and* (4.5) *respectively, and* $U_{N,ij}^{(k)} = T_{N,ij}^{(k)} + T_{N,i}^{(k)}\,T_{N,j}^{(k)}$ *have asymptotically a* cp(p+3)/2-*variate normal distribution with null mean vector and finite covariance matrix.*

The proof of this theorem is given in the Appendix A.

We may remark that the asymptotic normality of the random variables $N^{1/2}(T_{N,i}^{(k)} - \mu_{N,i}^{(k)}\ ;\ i = 1,\ldots,p;\ k = 1,\ldots,c)$ follows under slightly weaker conditions [cf. Puri and Sen (1966, Theorem 5.1)]. On the other hand the assumptions

I to V are needed for the asymptotic normality of

$$N^{1/2}\left[U_{N,ij}^{(k)} - \mu_{N,ij}^{(k)} - \mu_{N,i}^{(k)}\,\mu_{N,j}^{(k)}\right],\ i \leqq j = 1,\ldots,p;\ k=1,\ldots,c.$$

We shall now obtain the limiting distributions of $\mathcal{L}_N$ given by (3.24) under the null hypothesis $H_o^{(1)}$ and under a sequence of specific dispersion alternatives to be stated below. In what follows, we assume that for all $k = 1,\ldots,c$, $\lim_{N \to \infty} \lambda_N^{(k)} = \lambda^{(k)}$ exists and is positive.

Now under $H_o^{(1)}$, it follows from the results of Puri and Sen (1966) that $|T_{N,i}^{(k)}| = O_p(N^{-1/2})$ for all $i = 1,\ldots,p$ and $k = 1,\ldots,c$; and following routine but lengthy computations one can show that under $H_o^{(1)}$, asymptotically

$$N \operatorname{cov}\left\{S_{N,ij}^{(k)},\ S_{N,rs}^{(q)}\right\} \longrightarrow \left\{\frac{\delta_{kq}}{\lambda^{(k)}} - 1\right\} \nu_{ij,rs} \tag{5.1}$$

for all $i, j, r, s = 1,\ldots,p$; $k, q = 1,\ldots,c$, where δ_{kq} is the usual Kronecker delta and ν_{ijrs} is defined as in (4.7) with the further simplification that $F_{(i)}^{(k)} \equiv H_{(i)}$, $F_{(i,j,r,s)}^{(k)} \equiv H_{(i,j,r,s)}$ for all $k = 1,\ldots,c$; $i, j, r, s = 1,\ldots,p$. Consequently, using Theorems 4.1, 5.1 and some routine analysis, it can be shown that under $H_o^{(1)}$, $\mathcal{L}_N$ has asymptotically (unconditionally) a chi-square distribution with $(c-1)p(p+1)/2$ degrees of freedom.

We now formulate a class of dispersion alternatives.

For the scalar alternatives we let (as in Puri and Sen (1966))

$$F^{(k)}_{(i)}(x) = F_{(i)}([x - \mu_i]\delta^{(k)}_i);\ i = 1,\ldots,p;\ k=1,\ldots,c \quad (5.2)$$

where under $H^{(1)}_o$, all $\delta^{(k)}_i$ are equal to unity. Next, as in Sibuya (1959), we consider bivariate dependence functions $\Omega^{(k)}_{ij}$ defined as

$$\Omega^{(k)}_{ij} = F^{(k)}_{(i,j)}(x,y) \,/\, F^{(k)}_{(i)}(x)\ F^{(k)}_{(j)}(y) \qquad (5.3)$$

for all $i \neq j = 1,\ldots,p$; $k = 1,\ldots,c$. Clearly, $\Omega^{(k)}_{ij}$ is a function of $F^{(k)}_{(i)}$ and $F^{(k)}_{(j)}$, and will be equal to unity if $X^{(k)}_{i\alpha}$ and $X^{(k)}_{j\alpha}$ are statistically independent. Various properties of such dependence functions have been studied by Sibuya (1959) and others, and in nonparametric field, they appear to have some advantages over the other measures. We denote by Ω_{ij}, the same function (i.e. (5.3)) for the bivariate cdf $F_{(i,j)}(x,y)$, and relate

$$\Omega^{(k)}_{ij} = \Omega_{ij} + \omega^{(k)}_{ij} \quad \text{for } i \neq j = 1,\ldots,p;\ k=1,\ldots,c \quad (5.4)$$

(5.2), (5.3) and (5.4) together constitute our desired class of alternatives. Now we can rewrite (2.8) as

$$\theta_{ij}^{(k)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[F_{(i,j)}^{(k)}(x,y) - F_{(i)}^{(k)}(x)\, F_{(j)}^{(k)}(y)\right]$$

$$J'_{(i)}\left[H_{(i)}(x)\right] J'_{(j)}\left[H_{(j)}(y)\right] dH_{(i)}(x)\, dH_{(j)}(y) \quad (5.5)$$

for all i, $j = 1,\ldots,p$; $k = 1,\ldots,c$. From (5.3), (5.4), we note that $\theta_{ij}^{(k)}$ measures in a sense some functional of the dependence function $\Omega_{ij}^{(k)}$, for all i, $j = 1,\ldots,p$, and $k = 1,\ldots,c$.

For the study of the asymptotic non-null distribution of $\mathcal{L}_N$, we shall consider a sequence of alternatives converging to the null hypothesis in such a manner that the power of the test based on $\mathcal{L}_N$ lies in the open interval $(\varepsilon,1)$. Thus we replace $\delta_i^{(k)}$ in (5.2) by a sequence $\{1 + N^{-1/2}\beta_i^{(k)}\}$ where $\beta_i^{(k)}$, $i = 1,\ldots,p$; $k = 1,\ldots,c$ are real finite constants. Also, we replace (5.5) by

$$F_{(i,j)}^{(k)}(x,y) - F_{(i)}^{(k)}(x)\, F_{(j)}^{(k)}(y) =$$

$$\left[H_{(i,j)}(x,y) - H_{(i)}(x)\, H_{(j)}(y)\right]$$

$$+ N^{-1/2}\, \xi_{ij}^{(k)}(x,y) \quad (5.6)$$

for all $i \neq j = 1,\ldots,p$; $k = 1,\ldots,c$ where $\xi_{ij}^{(k)}(x,y)$ are real and finite valued functions. We shall denote such a a sequence of alternative hypotheses by $\{H_N\}$. Thus, on defining

$$\zeta_{ij}^{(k)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi_{ij}^{(k)}(x,y)\ J_{(i)}^{'}\left[H_{(i)}(x)\right] J_{(j)}^{'}\left|H_{(j)}(y)\right| dH_{(i)}(x)dH_{(j)}(y)$$

$$\text{for } i,\ j = 1,\ldots,p;\ k = 1,\ldots,c; \qquad (5.7)$$

$$\underset{\sim}{\zeta} = \left\{\zeta_{ij}^{(k)},\ i \leqq j = 1,\ldots,p\right\};\ k = 1,\ldots,c\ , \qquad (5.9)$$

and following some routine analysis, we arrive at the following theorem.

THEOREM 5.2. Under $\{H_N\}$, $\mathcal{L}_N$ has asymptotically a non-central chi-square distribution with $(c-1)p(p+1)/2$ degrees of freedom and the non centrality parameter

$$\Lambda = \sum_{k=1}^{c} \lambda_k \left[\underset{\sim}{\zeta}^{(k)}\ \underset{\sim}{\nu}^{*-1}\ \underset{\sim}{\zeta}^{(k)'}\right] \qquad (5.9)$$

where $\underset{\sim}{\nu}^{*-1}$ is the inverse of $\underset{\sim}{\nu}^{*}$, defined in (4.8).

In the parametric case, the likelihood ratio (L.R.) test for this problem is given in Anderson's (1958) book. He has only considered the distribution of the L.R. criterian under null hypothesis. However, on using Wald's (1943) methods, one can readily find out the asymptotic non-null distribution of the L. R. criterian under $\{H_N\}$. If $\underset{\sim}{\Sigma}^{(k)}$, $k = 1,\dots,c$ be the c dispersion matrices, under $\{H_N\}$, we will have

$$\underset{\sim}{\Sigma}^{(k)} = \underset{\sim}{\Sigma}_N^{(k)} = \underset{\sim}{\Sigma} + N^{1/2}\, \underset{\sim}{T}^{(k)} , \quad k = 1,\dots,c \tag{5.10}$$

where $\underset{\sim}{T}^{(k)}$, $k = 1,\dots,c$ are all real and finite. Now defining

$$\gamma_{ij,i'j'} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i\, x_j\, x_{i'}\, x_{j'}\, dF_{(i,j,i',j')}(x_i,x_j,x_{i'},x_{j'}) - \sigma_{ij}\,\sigma_{i'j'} \tag{5.11}$$

and adopting the suffixing system of (3.20), we write

$$\underset{\sim}{\Gamma} = ((\gamma_{rs}))_{r,s=1,\dots,p(p+1)/2} \tag{5.12}$$

$$\underset{\sim}{T}_o^{(k)} = \{\tau_{ij}^{(k)},\ i \leqq j = 1,\dots,p\},\ k = 1,\dots,c.$$

Then, we have the following theorem, the proof of which is omitted.

THEOREM 5.3. Under (5.10), the L. R. test criterian has asymptotically the non-central chisquare distribution with $(c-1)p(p+1)/2$ degrees of freedom, and the non-centrality parameter

$$\Delta = \sum_{k=1}^{c} \lambda_k \left[\underset{\sim}{T}_o^{(k)} \underset{\sim}{\Gamma}^{-1} \underset{\sim}{T}_o^{(k)} \right] \qquad (5.13)$$

Comparison of (5.9) and (5.13) will yield the asymptotic relative efficiency of the $\mathcal{L}_N$ test with respect to the L. R. test. Evidently, the same depends on (i) $J_i(u)$, $i = 1,\ldots,p$; (ii) $\{H_N\}$ and (iii) the unknown cdf $F(\underset{\sim}{x})$.

Among the possible members of the $\mathcal{L}_N$ tests, the two most appealing are (i) the Spearman type of test for which $J_{(i)}(u) = \sqrt{12}\,(u - 1/2)$, and (ii) the normal scores test for which $J_{(i)}(u)$ is the expected value of the ith order statistic of a sample of size N from a standardized normal cdf. For the normal scores test, it can be shown that if $F(\underset{\sim}{x})$is also normal, than (5.9) and (5.13) are equal, that is, the two tests are asymptotically power equivalent. The detailed study of the asymptotic relative efficiency is not studied here because the same depends on the unknown $F(\underset{\sim}{x})$ and $\{H_N\}$.

6a. NONPARAMETRIC TESTS FOR $H_o^{(2)}$

Under $H_o^{(2)}$ defined in (2.21) and (2.22), the basic permutation argument of Section 3, is no longer tenable, and so, it is difficult to derive the permutationally distribution free tests. However, we shall derive a class of rank order tests by centering the observations at the respective estimates of the location parameters (as is generally done in the univariate scale problems, see for example, Sukhatme (1958), Raghavachari (1965)), and working with these centered observations. Due to centering, the independence is vitiated, and as a result, one is faced with the question: Under what conditions the test based on the centered observations is asymptotically equivalent to the test based on the observations centered at the true locations? In this section we study this problem. For this purpose, we assume that $F^{(k)}(\underset{\sim}{x})$ is absolutely continuous having the continuous density function $f^{(k)}(\underset{\sim}{x})$; $k = 1,\ldots,c$, and the following conditions hold:

A. The densities $f^{(k)}(\underset{\sim}{x})$ of $F^{(k)}(\underset{\sim}{x})$, $k = 1,\ldots,c$ are diagonally symmetric about their location parameters.

B. $J_{(i)}(u) = \psi^{-1}(u)$ where ψ is symmetric about zero.

C. The functions $\frac{d}{dx} J_{(i)}\left[F^{(k)}_{(i)}(x)\right]$ are bounded for each $i = 1,\ldots,p$ and $k = 1,\ldots,c$.

Let now $\hat{\delta}^{(k)}_i$ be a consistent estimator of $\delta^{(k)}_i$ defined in (2.21), and denote

$$X^{*(k)}_{i\alpha} = X^{(k)}_{i\alpha} - \hat{\delta}^{(k)}_i \;:\; i = 1,\ldots,p; \quad k = 1,\ldots,c;\ \alpha = 1,\ldots,n_k\ . \tag{6.1}$$

In addition to the notations of the previous sections, we introduce the following:

$$F^{*(k)}_{N(i)}(x) = \frac{1}{n_k} \Big[\text{number of } X^{(k)}_{i\alpha} \text{ such that } (X^{(k)}_{i\alpha} - \hat{\delta}^{(k)}_{i})$$

$$\leqq x;\ \alpha = 1,\ldots,n_k\Big] \quad i = 1,\ldots,p;\ k = 1,\ldots,c \quad (6.2)$$

$$H^{*}_{N(i)}(x) = \sum_{k=1}^{c} \lambda^{(k)}_{N}\ F^{*(k)}_{N(i)}\ ,\ i = 1,\ldots,p \qquad (6.3)$$

$$F^{*(k)}_{N(i,j)}(x) = \frac{1}{n_k} \Big[\text{number of } (X^{(k)}_{i\alpha},\ X^{(k)}_{j\alpha}) \text{ such that}$$

$$(X^{(k)}_{i\alpha} - \hat{\delta}^{(k)}_{i},\ X^{(k)}_{j\alpha} - \delta^{(k)}_{j})$$

$$\leqq (x,y),\ \alpha = 1,\ldots,n_k\Big] \quad i,\ j = 1,\ldots,p\ . \quad (6.4)$$

$$H^{*}_{N(i,j)} = \sum_{k=1}^{c} \lambda^{(k)}_{N}\ F^{*(k)}_{N(i,j)}\ . \qquad (6.5)$$

Now consider the statistics

$$U^{*(k)}_{N(i,j)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{N(i)}\Big[H^{*}_{N(i)}(x_i)\Big]\ J_{N(j)}\ \Big[H^{*}_{N(j)}(x_j)\Big]\ dF^{*(k)}_{N(i,j)}$$

$$i \neq j = 1,\ldots,p;\ k = 1,\ldots,c \quad (6.6)$$

$$T^{*(k)}_{N,i} = \int_{-\infty}^{\infty} J_{N(i)}\Big[H^{*}_{N(i)}(x_i)\Big]\ dF^{(k)}_{N(i)}(x_i);\ i = 1,\ldots,p;$$

$$k = 1,\ldots,c\ . \quad (6.7)$$

Finally, concerning $\hat{\delta}^{(k)}_{i}$'s, we shall assume that

$$N^{1/2}\,|\hat{\delta}_i^{(k)} - \delta_i^{(k)}| = \mathcal{O}_p(1) \text{ for all } i = 1,\ldots,p;$$

$$k = 1,\ldots,c. \qquad (6.8)$$

Then the main theorem of this section is the following.

THEOREM 6.1. Under the assumptions A, B *and* C, *the random variables* $N^{1/2}[(T_{N,ij}^{(k)} - \mu_{N,ij}^{(k)}),\ (T_{N,i}^{(k)} - \mu_{N,i}^{(k)})\ i \leqq j = 1,\ldots,p$ $k = 1,\ldots,c]$ *and* $N^{1/2}[(U_{N,ij}^{(k)} - \mu_{N,ij}^{(k)}),\ (T_{N,i}^{*(k)} - \mu_{N,i}^{(k)}),$ $i \leqq j = 1,\ldots,p,\ k = 1,\ldots,c]$ *have the same limiting distribution*.

The proof is sketched in the Appendix.

It therefore follows from Theorem 6.1 that if we work with observations centered at the respective estimated location parameters, the test considered in Section 4 will be asymptotically a distribution free test for $H_o^{(2)}$, and will have the same asymptotic properties.

6b. TESTS FOR $H_o^{(3)}$

In an earlier paper (1966), the authors had considered the problem of testing the equality of location parameters under the assumption that the covariance matrices were the same. The tests of this assumption (that the covariance matrices are same) are already considered in the previous sections. The purpose of this section is to sketch very briefly the problem of testing simultaneously the identity of location parameters and covariance matrices. Here also since the null hypothesis reduces to the identity of the c cdfs, the basic permutation arguments of Section 3 will remain valid. Here, we shall have to work with the set of statistics $\{T_{N,ij}^{(k)}$ and $T_{N,i}^{(k)}$ for $i \leqq j = 1,\ldots,p$, $k = 1,\ldots,c\}$. It is therefore equivalent to work with the set of statistics,

$$U_{N,ij}^{(k)} = \frac{1}{n_k} \sum_{\alpha=1}^{n_k} E_{N,R_{i\alpha}^{(k)}}^{(i)} E_{N,R_{j\alpha}^{(k)}}^{(j)} , \tag{6.9}$$

for $k = 1,\ldots,c$, $i \leqq j = 1,\ldots,p$ and

$$T_{N,i}^{(k)} = \frac{1}{n_k} \sum_{\alpha=1}^{n_k} E_{N,R_{i\alpha}}^{(i)} \quad \text{for } i = 1,\ldots,p,\ k = 1,\ldots,c. \tag{6.10}$$

The permutation covariance matrix of these $cp(p+3)/2$ random variables can be found in a similar manner and a test-statistic similar to $\pounds_N$ in (3.24) can be constructed. Finally, Theorem 5.1 may be used to study the asymptotic properties of this test.

7a. APPENDIX A

Proof of Theorem 5.1. Writing

$$\begin{aligned} J_{N(i)}\left[H_{N(i)}\right] J_{N(j)}\left[H_{N(j)}\right] = \Big\{ & J_{N(i)}\left[H_{N(i)}\right] J_{N(j)}\left[H_{N(j)}\right] \\ & - J_{(i)}\left[H_{N(i)}\right] J_{(j)}\left[H_{N(j)}\right]\Big\} \\ & + J_{(i)}\left[H_{N(i)}\right] J_{(j)}\left[H_{N(j)}\right] , \qquad (7.1) \end{aligned}$$

$$\begin{aligned} & J_{(i)}\left[H_{N(i)}\right] J_{(j)}\left[H_{N(j)}\right] \\ & = J_{(i)}\left[H_{(i)}\right] J_{(j)}\left[H_{(j)}\right] + (H_{N(i)} - H_{(i)}) J'_{(i)}\left[H_{(i)}\right] J_{(j)}\left[H_{(j)}\right] \\ & + (H_{N(j)} - H_{(j)}) J'_{(j)}\left[H_{(j)}\right] J_{(i)}\left[H_{(i)}\right] \\ & + \frac{1}{2} (H_{N(i)} - H_{(i)})^2 J''_{(i)}\left[\theta H_{N(i)} + (1 - \theta) H_{(i)}\right] \\ & \qquad J_{(j)}\left[\theta H_{N(j)} + (1 - \theta) H_{(j)}\right] \\ & + \frac{1}{2} (H_{N(j)} - H_{(j)})^2 J''_{(j)}\left[\theta H_{N(j)} + (1 - \theta) H_{(j)}\right] \\ & \qquad J_{(i)}\left[\theta H_{N(i)} + (1 - \theta) H_{(i)}\right] \\ & + (H_{N(i)} - H_{(i)})(H_{N(j)} - H_{(j)}) J'_{(i)}\left[\theta H_{N(i)} + (1 - \theta) H_{(i)}\right] \\ & \qquad J'_{(j)}\left[\theta H_{N(j)} + (1 - \theta) H_{(j)}\right] \qquad (7.2) \end{aligned}$$

where $0 < \theta < 1$ and

$$dF^{(k)}_{n_k[i,j]} = d(F^{(k)}_{n_k[i,j]} - F^{(k)}_{[i,j]}) + dF^{(k)}_{[i,j]} \qquad (7.3)$$

and, proceeding as in Bhuchongkul (1964) and Puri and Sen (1966), we can express

$$U^{(k)}_{N,ij} = A^{(k)}_{1N(i,j)} + A^{(k)}_{2N(i,j)} + A^{(k)}_{3N(i,j)} + \sum_{r=1}^{8} \mathcal{C}^{(k)}_{rN(i,j)} \qquad (7.4)$$

and

$$T^{(k)}_{N,i} = \mu^{(k)}_{N(i)} + B^{(k)}_{1N(i)} + B^{(k)}_{2N(i)} + \sum_{r=1}^{4} \mathcal{C}^{(k)}_{rN(i)} \qquad (7.5)$$

where

$$A^{(k)}_{1N(i,j)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{(i)}\left[H_{(i)}(x_i)\right] dF^{(k)}_{n_k(i,j)}(x_i,x_j) \qquad (7.6)$$

$$A^{(k)}_{2N(i,j)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[H_{N(i)}(x_i) - H_{(i)}(x_i)\right] J'_{(i)}\left[H_{(i)}(x_i)\right] J_{(j)}\left[H_{(j)}(x_j)\right] dF^{(k)}_{[i,j]}(x_i,x_j) \qquad (7.7)$$

$$A^{(k)}_{3N(i,j)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[H_{N(j)}(x_j) - H_{(j)}(x_j)\right] J'_{(j)}\left[H_{(j)}(x_j)\right] J_{(i)}\left[H_{(i)}(x_i)\right] dF^{(k)}_{[i,j]}(x_i,x_j) \qquad (7.8)$$

$$B^{(k)}_{1N(i)} = \int_{-\infty}^{\infty} J_{(i)}\left[H_{(i)}(x_i)\right] d\left[F^{(k)}_{N(i)}(x_i) - F^{(k)}_{(i)}(x_i)\right] \qquad (7.9)$$

$$B^{(k)}_{2N(i)} = \int_{-\infty}^{\infty} \left[H_{N(i)}(x_i) - H_{(i)}(x_i)\right] J'_{(i)} \left[H_{(i)}(x_i)\right] dF^{(k)}_{(i)}(x_i) \quad (7.10)$$

$$\mathcal{B}^{(k)}_{1N(i,j)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[H_{N(i)} - H_{(i)}\right] J'_{(i)} \left[H_{(i)}\right] J_{(j)} \left[H_{(j)}\right] d\left[F^{(k)}_{N(i,j)} - F^{(k)}_{(i,j)}\right] \quad (7.11)$$

$$\mathcal{B}^{(k)}_{2N(i,j)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[H_{N(j)} - H_{(j)}\right] J'_{(j)} \left[H_{(j)}\right] J_{(i)} \left[H_{(i)}\right] d\left[F^{(k)}_{N(i,j)} - F^{(k)}_{(i,j)}\right] \quad (7.12)$$

$$\mathcal{B}^{(k)}_{3N(i,j)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2} \left[H_{N(i)} - H_{(i)}\right]^2 J''_{(i)} \left[\theta H_{N(i)} + (1-\theta)H_{(i)}\right] J_{(j)} \left[\theta H_{N(j)} + (1-\theta)H_{(j)}\right] dF^{(k)}_{N(i,j)} \quad (7.13)$$

$$\mathcal{B}^{(k)}_{4N(i,j)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2} \left[H_{N(j)} - H_{(j)}\right]^2 J''_{(j)} \left[\theta H_{N(j)} + (1-\theta)H_{(j)}\right] J_{(i)} \left[\theta H_{N(i)} + (1-\theta)H_{(i)}\right] \quad (7.14)$$

$$\mathcal{B}^{(k)}_{5N(i,j)} = \iint_{I_{N(i)} \cap I_{N(j)}} \left|H_{N(i)} - H_{(i)}\right| \left|H_{N(j)} - H_{(j)}\right| J'_{(i)} \left[\theta H_{N(i)} + (1-\theta)H_{(i)}\right] J'_{(j)} \left[\theta H_{N(j)} + (1-\theta)H_{(j)}\right] dF^{(k)}_{N(i,j)} \quad (7.15)$$

$$\mathcal{B}^{(k)}_{6N(i,j)} = \iint\limits_{R\times R - I_{N(i)}\cap I_{N(j)}} \Big[J_{(i)}\,[H_{(i)}]\, J_{(j)}\,[H_{(j)}] + (H_{N(i)} - H_{(i)})\, J'_{(i)}\,[H_{(i)}]\, J_{(j)}\,[H_{(j)}] + (H_{N(j)} - H_{(j)})\, J_{(i)}\,[H_{(i)}]\, J'_{(j)}\,[H_{(j)}]\Big]\, dF^{(k)}_{(i,j)} \qquad (7.16)$$

$$\mathcal{B}^{(k)}_{7N(i,j)} = \iint\limits_{R\times R - I_{N(i)}\cap I_{N(j)}} J_{N(i)}\,\Big[H_{N(i)}\Big]\, J_{N(j)}\,\Big[H_{N(j)}\Big]\, dF^{(k)}_{N(i,j)} \qquad (7.17)$$

$$\mathcal{B}^{(k)}_{8N(i,j)} = \iint\limits_{I_{N(i)}\cap I_{N(j)}} \Big[J_{N(i)}\,[H_{N(i)}]\, J_{N(j)}\,[H_{N(j)}] - J_{(i)}\,[H_{N(i)}]\, J_{(j)}\,[H_{N(j)}]\Big]\, dF^{(k)}_{N(i,j)} \qquad (7.18)$$

$$\mathcal{C}^{(k)}_{1N(i)} = -(N+1)^{-1} \int_{-\infty}^{\infty} H_{N(i)}\, J'_{(i)}[H_{(i)}]\, dF^{(k)}_{N(i)} \qquad (7.19)$$

$$\mathcal{C}^{(k)}_{2N(i)} = \int_{-\infty}^{\infty} (H_{N(i)} - H_{(i)})\, J'_{(i)}[H_{(i)}]\, d[F^{(k)}_{N(i)} - F^{(k)}_{(i)}] \qquad (7.20)$$

$$\mathcal{C}^{(k)}_{3N(i)} = \int_{-\infty}^{\infty} [J_{(i)}[\tfrac{N}{N+1} H_{N(i)}] - J_{(i)}[H_{(i)}] - (\tfrac{N}{N+1} H_{N(i)} - H_{(i)})J'_{(i)}[H_{(i)}]]\,dF^{(k)}_{N(i)} \qquad (7.21)$$

$$\mathcal{C}^{(k)}_{4N(i)} = \int_{-\infty}^{\infty} [J_{N(i)}[\tfrac{N}{N+1} H_{N(i)}] - J_{(i)}[\tfrac{N}{N+1} H_{N(i)}]]\,dF^{(k)}_{N(i)} \qquad (7.22)$$

Now proceeding precisely as in Bhuchongkul (1964) and Puri and Sen (1966), it can be easily shown that the $\mathcal{B}$ and $\mathcal{C}$ terms are all $o_p(N^{-1/2})$. Thus

$N^{1/2}[\, U^{(k)}_{N,ij} - \sum_{h=1}^{3} A^{(k)}_{hN(i,j)}]$ as well as $N^{1/2}[T^{(k)}_{N(i)} - \mu^{(k)}_{N(i)}] -$

$N^{1/2}[B^{(k)}_{1N(i)} + B^{(k)}_{2N(i)}]$ tends to zero in probability

as $N \to \infty$, for each $i,j = 1,\ldots,p$ and $k = 1,\ldots,c$.

Hence it suffices to establish the joint asymptotic normality of

$$N^{1/2}\left[\sum_{h=1}^{3} A^{(k)}_{hN(i,j)};\ 1 \leqq i < j \leqq p;\ k = 1,\ldots,c\right] \text{ and}$$

$$N^{1/2}\left[B^{(k)}_{1N(i)} + B^{(k)}_{2N(i)};\ i = 1,\ldots,p;\ k = 1,\ldots,c\right].$$

Now let us denote

$$U^{(i,j)}_{1n_k,\alpha}(X^{(k)}_{i\alpha}, X^{(k)}_{j\alpha}) = J_{(i)}\left[H_{(i)}(X^{(k)}_{i\alpha})\right] J_{(j)}\left[H_{(j)}(X^{(k)}_{j\alpha})\right] \quad (7.23)$$

$$U^{(i,j)}_{2n_{k,r,\alpha}}(X^{(r)}_{i\alpha}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[F^{(r)}_{1X_{i\alpha}}(x_i) - F^{(r)}_{i}(x_i)\right] \cdot J'_{(i)}[H_{(i)}]\, J_{(j)}[H_{(j)}]\, dF^{(k)}_{(i,j)}(x_i, x_j) \quad (7.24)$$

where

$$F^{(r)}_{1X_{i\alpha}}(x_i) = 1 \text{ if } X^{(r)}_{i\alpha} \leqq x_i \text{ and is zero otherwise.} \quad (7.25)$$

$$U^{(i,j)}_{3n_{k,r,\alpha}}(X^{(r)}_{j\alpha}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[F^{(r)}_{1X_{j\alpha}}(x_j) - F^{(r)}_{(j)}(x_j)\right] \cdot J'_{(j)}[H_{(j)}]\, J_{(i)}[H_{(i)}]\, dF^{(k)}_{(i,j)} \quad (7.26)$$

where

$$F^{(r)}_{1X_{j\alpha}}(x_j) = 1 \text{ if } X^{(r)}_{j\alpha} \leqq x_j \text{ and is zero otherwise.} \quad (7.27)$$

Then, we can write

$$\sum_{h=1}^{3} A_{hN(i,j)}^{(k)} = \sum_{\substack{r=1\\ r\neq k}}^{c} \Big[\lambda_N^{(r)} \cdot \frac{1}{n_r} \sum_{\alpha=1}^{n_r} \Big\{U_{2n_k,r,\alpha}(X_{i\alpha}^{(r)}) + U_{3n_k,r,\alpha}^{(i,j)}(X_{j\alpha}^{(r)})\Big\}\Big] + \frac{1}{n_k} \sum_{\alpha=1}^{n_k} \Big[U_{1n_k,\alpha}^{(i,j)}(X_{i\alpha}^{(k)}, X_{j\alpha}^{(k)}) + \lambda_N^{(k)} \Big\{U_{2n_k,k,\alpha}(X_{i\alpha}^{(k)}) + U_{3n_k,k,\alpha}(X_{j\alpha}^{(k)})\Big\}\Big] \tag{7.28}$$

$$B_{1N(i)}^{(k)} + B_{2N(i)}^{(k)} = - \sum_{\substack{r=1\\ r\neq k}}^{c} \Big[\lambda_N^{(r)} \cdot \frac{1}{n_r} \sum_{\alpha=1}^{n_r} \Big\{B_i(X_{i\alpha}^{(r)}) - E\,B_i(X_{i\alpha}^{(r)})\Big\}\Big] + \frac{1}{n_k} \sum_{\alpha=1}^{n_k} \Big\{J_{(i)}\Big[H_{(i)}(X_{i\alpha}^{(k)})\Big] - \lambda_N^{(k)} B_i(X_{i\alpha}^{(k)}) - E\Big[J_{(i)}\Big[H_{(i)}(X_{i\alpha}^{(k)})\Big] - \lambda_N^{(k)} B_i(X_{i\alpha}^{(k)})\Big]\Big\} \tag{7.29}$$

where

$$B_i(x_i^{(r)}) = \int_{x_{oi}}^{x_i(r)} J'_{(i)}\Big[H_{(i)}(y)\Big]\, dF_{(i)}^{(r)}(y) \tag{7.30}$$

with x_{oi} determined somewhat arbitrary, say by $H_{(i)}(x_{oi}) = 1/2$; E represents the expectation and $X_{i\alpha}^{(k)}$, $\alpha = 1,\ldots,n_k$ has the distribution $F_i^{(k)}$; $i = 1,\ldots,p$; $k = 1,\ldots,c$.

Hence with $d_{ij}^{(k)}$ and $e_i^{(k)}$, $1 \leqq i \leqq j \leqq p$, $k = 1,\ldots,c$ any real constants, not all zero, we can write

$$N^{1/2}\Big[\sum_{i\leqq j=1}^{p} \sum_{k=1}^{c} d_{ij}^{(k)} \Big\{A_{1N(i,j)}^{(k)} + A_{2N(i,j)}^{(k)} + A_{3N(i,j)}^{(k)}\Big\} + \sum_{i=1}^{p} \sum_{k=1}^{c} e_i^{(k)} \Big\{B_{1N(i)}^{(k)} + B_{2N(i)}^{(k)}\Big\}\Big]$$

$$= N^{1/2} \sum_{\substack{r=1\\ r\neq k}}^{c} \lambda_N^{(r)} \Big[\frac{1}{n_r} \sum_{\alpha=1}^{n_r} \Big[\sum_{i\leqq j} d_{ij}^{(k)} \Big\{U_{2n_k,r,\alpha}(X_{i\alpha}^{(e)}) +$$

$$U_{3n_k,r,\alpha}(X_{j\alpha}^{(r)})\Big\} \sum_{i=1}^{p} e_i^{(k)}\Big\{B_i(X_{i\alpha}^{(r)}) - E\, B_i(X_{i\alpha}^{(r)})\Big\}\Big]\Big]$$

$$+ N^{1/2} \frac{1}{n_k} \sum_{\alpha=1}^{n_k} \Big[\sum_{i\leq j} d_{ij}^{(k)} \Big\{ U^{(i,j)}_{1n_k,\alpha} (X_{i\alpha}^{(k)},\, X_{j\alpha}^{(k)}) + \lambda_N^{(k)} U_{2n_k,k,\alpha}(x_{i\alpha}^{(k)})$$

$$+ \lambda_N^{(k)}\, U_{3n_k,k,\alpha}\, (X_{j\alpha}^{(k)})\Big\} - \sum_{i=1}^{p} e_i^{(k)} \Big\{J_{(i)}\Big[H_{(i)}(x_{i\alpha}^{(k)})\Big]$$

$$- \lambda_N^{(k)}\, B_i(X_{i\alpha}^{(k)}) - E\Big\{J_{(i)}\Big[H_{(i)}(X_{i\alpha}^{(k)})\Big] - \lambda_N^{(k)}\, B_i(X_{i\alpha}^{(k)})\Big\}\Big]. \qquad (7.31)$$

These are the c-summations. They involve independent samples of identically distributed random variables, each having finite third moment. The proof follows.

To compute the variance covariance terms, we note using (7.24) to (7.29) that

$$N \operatorname{var} \Big[\sum_{h=1}^{3} A_{hN(i,j)}^{(k)} \Big] = \sum_{\substack{r=1 \\ r\neq k}}^{c} \lambda_N^{(r)} \operatorname{var} \Big\{ U^{(i,j)}_{2n_k,r,\alpha}(X_{i\alpha}^{(r)}) +$$

$$U^{(i,j)}_{3n_k,r,\alpha}\, (X_{j\alpha}^{(r)})\Big\} + \frac{1}{\lambda_N^{(k)}} \operatorname{var} \Big\{ U^{(i,j)}_{1n_k,\alpha}(X_{i\alpha}^{(k)}, X_{j\alpha}^{(k)}) +$$

$$\lambda_N^{(k)}\, U^{(i,j)}_{2n_k,k,\alpha}(X_{i\alpha}^{(k)}) + \lambda_N^{(k)} U^{(i,j)}_{3n_k,k,\alpha}(X_{j\alpha}^{(k)})\Big\}. \qquad (7.32)$$

$$N \operatorname{cov} \Big(\sum_{h=1}^{3} A_{hN(i,j)}^{(k)},\ \sum_{h=1}^{3} A_{hN(r,s)}^{q} \Big)$$

$$= \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \operatorname{cov} \big(A_{\alpha N(i,j)}^{(k)},\ A_{\beta N(r,s)}^{(q)}\big)\ . \qquad (7.33)$$

The variance-covariance terms of $\{N^{1/2}(B_{1N(i)}^{(k)} + B_{2N(i)}^{(k)})$, $i = 1,\ldots,p;\ k = 1,\ldots,c$ are given in the expression (5.3) to (5.6) of Puri and Sen (1966). The other variance-covariance terms can be evaluated analogously.

7b. APPENDIX B

The proof of Theorem 6.1 rests on the following two lemmas the first of which is due to Raghavachari (1965) and the second is a slight generalization of Raghavachari (1965).

LEMMA 1. (Raghavachari.) If the assumption (c) holds, then $\frac{d}{dx} J\left[H_{(i)}(x)\right]$ is also bounded, where $H_{(i)}(x)$ is given by (2 3).

LEMMA 2. Let $(X_1, Y_1), \ldots, (X_N, Y_N)$ be i·i·d·r·vs with continuous cdf $F(x - \xi, y - \eta)$ where F is assumed to be symmetric about $(0,0)$. Let $F_1(x)$ and $F_2(y)$ be the marginal cdfs of X_i and Y_i respectively. Let $\hat{\xi}(X_1, \ldots X_N)$ and $\hat{\eta}(Y_1, \ldots, Y_N)$ be consistent estimators of ξ and η respectively such that $N^{1/2}(\hat{\xi} - \xi)$ as well as $N^{1/2}(\hat{\eta} - \eta)$ is bounded in probability as $N \longrightarrow \infty$. Assume that there exists a function $M(x,y) = M_1(x)M_2(y)$ such that the following conditions are satisfied.

(i) $E\left[M_1(x)\right]^2 < \infty \qquad E\left[M_2(y)\right]^2 < \infty$,

(ii) $M_1'(x) = \dfrac{\partial M_1(x)}{\partial x}$ and $M_2'(y) = \partial M_2(y)/\partial y$ exist ,

(iii) $\left|M_1'(x - t_1)\right| \leqq K_1 T_1(x)$, $\left|M_2'(y - t_2)\right| \leqq K\, T_2(y)$

uniformly in t_1 and t_2 for $|t_i| \leqq c_i$; $i = 1,2$; (c_i and K_i, $i = 1,2$ are constants).

(iv) $E\left[T_1(X)\right]^2 < \infty$, $E\left[T_2(Y)\right]^2 < \infty$, where X and Y have the distribution functions $F_1(x - \xi)$ and $F_2(y - \eta)$ respectively.

(v) $E\left[M_1'(x - \xi)\right] = 0$, $E\left[M_2'(y - \eta)\right] = 0$.

Then

$$Q_N = N^{-1/2} \sum_{i=1}^{N} \{M_1(x_i - \xi) \cdot M_2(y_i - \eta) - M_1(X_i - \hat{\xi}) \cdot M_2(y_i - \hat{\eta})\} \xrightarrow{p} 0$$

Proof of Lemma 2. Without loss of generality we can take $\xi = \eta = 0$. Then

$$Q_N = N^{-1/2} \sum_{i=1}^{N} \{M_1(X_i)\ M_2\ (Y_i) - M_1(X_i - \hat{\xi}) \cdot M_2(Y_i - \hat{\eta})\}$$

$$= - \frac{t_1}{N} \sum_{i=1}^{N} M_2(Y_i)\ M_1'(X_i - \frac{\theta_1 t_1}{\sqrt{N}} - \frac{t_2}{\sqrt{N}} \sum_{i=1}^{N} M_1(x_i)\ M_2'(Y_i - \frac{\theta_2 t_1}{\sqrt{N}}) \tag{7.34}$$

after neglecting terms of order O(1/N). Under the assumptions (i) to (iv), of the lemma, it is easy to note (by applying Tchebychev's inequality) that

$$\frac{t_1}{N} \sum_{i=1}^{N} \Big[M_2(Y_i)\ M_1'\ (X_i - \frac{\theta_1 t_1}{\sqrt{N}} - E\{M_2(Y_i)\ M_1'\ (X_i - \frac{\theta_1 t_1}{\sqrt{N}})\} \Big|^2 \longrightarrow 0\ . \tag{7.35}$$

Furthermore, by dominated convergence theorem,

$$E\{M_2(Y_i)\, M_1'\,(X_i - \frac{\theta_1 t_1}{\sqrt{N}}\} \longrightarrow E\left[M_1'(X_i)\, M_2\,(Y_i)\right] \qquad (7.36)$$

uniform in t_1, for $|t_1| \leqq c_1$; and since under the assumed conditions, $E\{M_1'(x_i)\, M_2(y_i)\} = 0$ it follows that

$$\sup_{|t_1| \leqq c_1} \left| \frac{t_1}{N} \sum_{i=1}^{N} M_2(Y_i)\, M_1'\,(X_i - \frac{\theta_1 t_1}{\sqrt{N}}\,) \right| \xrightarrow{P} 0\ .$$

Hence the first factor on the right hand side $\xrightarrow{P} 0$ as $N \longrightarrow \infty$. Similarly the second factor $\xrightarrow{P} 0$ as $N \longrightarrow \infty$. Hence $Q_N \xrightarrow{p} 0$ as $N \longrightarrow \infty$.

Proof of the theorem. Without any loss of generality we can assume that $\chi_k = 0$, $k = 1,\ldots,c$. The proof of the theorem is accomplished by showing that (a) $N^{1/2} (U^{*(k)}_{N,ij} - U^{(k)}_{N,ij}) \xrightarrow{p} 0$ as $N \longrightarrow \infty$ for each i, $j = 1,\ldots,p$ and $k = 1,\ldots,c$, and (b) $N^{1/2}(T^{*(k)}_{N(i)} - T^{(k)}_{N(i)}) \xrightarrow{p} 0$ as $N \longrightarrow \infty$ for each $i = 1,\ldots,p$, and $k = 1,\ldots,c$.

To prove (a), note that proceeding as in Theorem 1, we can rewrite $T^{*(k)}_{N(i,j)}$ as

$$U^{*(k)}_{N,ij} = A^{*(k)}_{N(i,j)} + A^{*(k)}_{2N(i,j)} + A^{*(k)}_{3N(i,j)} + \sum_{r=1}^{8} \mathcal{Z}^{*(k)}_{rN(i,j)}$$

$$A^{*(k)}_{1N(i,j)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{(i)} \left[H_{(i)}\right] J_{(j)} \left[H_{(j)}\right] dF^{*(k)}_{N(i,j)}$$

$$A^{*(k)}_{2N(i,j)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[H^{*}_{N(i)} - H_{(i)}\right] J'_{(i)} \left[H_{(i)}\right] J_{(j)} \left[H_{(j)}\right] dF^{(k)}_{(i,j)}$$

$$A^{*}_{3N(i,j)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[H^{*}_{N(j)} - H_{(j)}\right] J'_{(j)} \left[H_{(j)}\right] J_{(i)} \left[H_{(i)}\right] dF^{(k)}_{(i,j)}$$

and the $\mathcal{C}^{*}$ terms are all $o_p(N^{-1/2})$.

Proceeding again as in Theorem 1, we obtain

$$N^{1/2}(A^{*(k)}_{1N(i,j)} + A^{*(k)}_{2N(i,j)} + A^{*(k)}_{3N(i,j)}) -$$

$$N^{1/2}(A^{(k)}_{1N(i,j)} + A^{(k)}_{2N(i,j)} + A^{(k)}_{3N(i,j)})$$

$$= N^{1/2} \sum_{\substack{r=1 \\ r\neq k}}^{c} \Big[\lambda^{(k)}_{N} \cdot \frac{1}{n_r} \sum_{\alpha=1}^{n_r} \{[U^{(i,j)}_{2Nkr,\alpha}(X^{(r)}_{i\alpha} - \hat{\nu}^{(r)}_{i}) +$$

$$U^{(i,j)}_{3Nkr,\alpha}(X^{(r)}_{j\alpha} - \hat{\nu}^{(r)}_{j})] - [U^{(i,j)}_{2Nk,r}(X_{i\alpha}) + U^{(i,j)}_{3Nkr,\alpha}(X^{(r)}_{j\alpha})]\}\Big] +$$

$$N^{1/2} \frac{1}{n_k} \sum_{\alpha=1}^{n_k} \Big[U^{(i,j)}_{1Nk,\alpha}(X^{(k)}_{i\alpha} - \hat{\nu}^{(k)}_{j}, X^{(k)}_{j\alpha} - \hat{\nu}^{(k)}_{j}) +$$

$$\lambda^{(k)}_{N} \{U^{(i,j)}_{2Nkk,\alpha}(X^{(k)}_{i\alpha} - \hat{\nu}^{(k)}_{i}) + U^{(i,j)}_{3Nkk,\alpha}(X^{(k)}_{j\alpha} - \hat{\nu}^{(k)}_{j})\} -$$

$$U^{(i,j)}_{1Nk,\alpha}(X^{(k)}_{i\alpha}, X^{(k)}_{j\alpha}) - \lambda^{(k)}_{N}\{U^{(i,j)}_{2Nkk,\alpha}(X^{(k)}_{i\alpha}) + U^{(i,j)}_{3Nkk}(X^{(k)}_{j\alpha})\}\Big]$$

$$= \sum_{\substack{r=1 \\ r\neq k}}^{c} \left[L^{(1)}_{N,r} + L^{(2)}_{N,r}\right] + L^{(1)}_{N,k} + L^{(2)}_{N,k} + L^{(3)}_{N,k}$$

where

$$L^{(1)}_{N,r} = N^{-1/2} \sum_{\alpha=1}^{n_r} \left\{ \left[U^{(i,j)}_{2Nk,r,\alpha} (X^{(r)}_{i\alpha} - \nu^{(r)}_i) - U^{(i,j)}_{2Nk,r,\alpha} (X^{(r)}_{i\alpha}) \right] \right\}$$

$r=1,\ldots,c;\ r\neq k$

$$L^{(2)}_{N,r} = N^{-1/2} \sum_{\alpha=1}^{n_r} \left\{ \left[U^{(i,j)}_{3Nk,r,\alpha} (X^{(r)}_{j\alpha} - \nu^{(r)}_j) - U^{(i,j)}_{3Nk,r,\alpha} (X^{(r)}_{j\alpha}) \right] \right\}$$

$r=1,\ldots,c;\ r\neq k.$

$$L^{(1)}_{N,k} = N^{-1/2} \sum_{\alpha=1}^{n_k} \left\{ \left[U^{(i,j)}_{1N\alpha,k} (X^{(k)}_{i\alpha} - \hat{\nu}^{(k)}_i, X^{(k)}_{j\alpha} - \hat{\nu}^{(k)}_j) - U^{(i,j)}_{1N\alpha,k} (X^{(k)}_{i\alpha}, X^{(k)}_{j\alpha}) \right] \right\}$$

$$L^{(2)}_{N,k} = N^{-1/2} \sum_{\alpha=1}^{n_k} \left\{ U^{(i,j)}_{2Nkk,\alpha} (X^{(k)}_{i\alpha} - \hat{\nu}^{(k)}_i) - U^{(i,j)}_{2Nkk,\alpha} (X^{(k)}_{i\alpha}) \right\}$$

$$L^{(3)}_{N,k} = N^{-1/2} \sum_{\alpha=1}^{n_k} \left\{ U^{(i,j)}_{3Nkk,\alpha} (X^{(k)}_{j\alpha} - \hat{\nu}^{(k)}_j) - U^{(i,j)}_{3Nkk,\alpha} (X^{(k)}_{j\alpha}) \right\} .$$

Consider first

$$L^{(1)}_{N,r} = N^{-1/2} \sum_{\alpha=1}^{n_r} \iint \left[F^{(r)}_{1X_{i\alpha}} (X_i - \hat{\nu}^{(r)}_i) - F^{(r)}_{1X_{i\alpha}} (X_i) \right]$$

$$J'_{(i)} \left[H_i(x_i) \right] J_{(j)} \left[H_{(j)}(x_j) \right] dF^{(r)}_{(i,j)}(x_i, x_j) .$$

Denote

$$B_{(r)}(X_{i\alpha}) = \iint F^{(r)}_{1X_{i\alpha}}(x_i)\, J'_{(i)}\left[H_i(x_i)\right] J_{(j)}\left[H_j(x_j)\right]$$

$$f^{(r)}_{ij}(x_i, x_j)\,dx_i\,dx_j$$

$$\left|B'_{(r)}(x^{-1}_{i\alpha})\right| = \left|J'_{(i)}\left[H_i(x_i - t)\right]\right| \left|\int_{-\infty}^{\infty} J_j(H_j(x_j))f_j(x_j)\,dx_j\right|$$

$$\leqq K \left|J'_{(i)}\left[H_i(x_i - t)\right]\right| \leqq K' \text{ by virtue of Lemma 1.}$$

Furthermore $EB'_{(r)}(X_{i\alpha}) = 0$

Hence, by Lemma 2, $L^{(1)}_{N,r} \xrightarrow{p} 0$ as $N \longrightarrow \infty$

Similarly $L^{(2)}_{N}$, $L^{(2)}_{N,k}$, $L^{(3)}_{N,k}$ and $L^{(1)}_{N,k}$ all $\xrightarrow{p} 0$ as $N \longrightarrow \infty$.

Hence $N^{1/2}\left[U^{(k)}_{N(i,j)} - \mu^{(k)}_{N(i,j)};\ i, j = 1,\ldots,p;\ k = 1,\ldots,c\right]$

and $N^{1/2}\left[U^{(k)}_{N(i,j)} - \mu^{(k)}_{N(i,j)}\right]$ have the same limiting distributions. The proof of (b) is similar to that of Theorem 3.1 of Raghavachari (1965), and is therefore omitted.

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